EXCEPTIONAL VALUES OF ENTIRE FUNCTIONS OF FINITE ORDER IN ONE OF THE VARIABLES

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ABSTRACT. Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w with $n \geq 2$. Let Ω be the set of points $z \in \mathbb{C}^n$ where $F(z,w)$ is a non-constant function omitting a value $\pi(z)$. Near a finite accumulation point z_0 of Ω, we prove in the main result (Theorem 1) that Ω is a local analytic set and $\pi(z)$ is holomorphic, and show the existence of a proper globally analytic set Δ of $\mathbb{C}^n$ such that either $\Omega \subset \Delta$ or $\Omega = \mathbb{C}^n \setminus \Delta$, being possible in the last case to also determine $F(z,w)$ in terms of $\pi(z)$. We apply this result to several problems. First, we extend a Theorem due to Nishino about exceptional values when near z_0 dimension of Ω is n and assure the existence of a meromorphic function $\alpha(z)$ in $\mathbb{C}^n$ such that $\pi(z) = \alpha(z)$ except at points where $\alpha(z)$ has poles or $F(z,w)$ is constant (also being $F(z,w)$ a polynomial in w if $\alpha(z)$ is ∞). After, we prove that Ω is a local analytic set in $\mathbb{C}^n$ and the existence of a proper analytic subset E of $\mathbb{C}^n$ such that $\Omega \subset E$ or $\Omega = \mathbb{C}^n \setminus E$. Finally, we generalize a Lelong-Gruman Theorem about the set of points z where $\pi(z) = 0$.

CONTENTS

1. INTRODUCTION

Consider the product space $\mathbb{C}^n \times \mathbb{C}$ of $n+1$ variables $z_1, \ldots, z_n, w$, where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $w \in \mathbb{C}$. Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$. Given $z \in \mathbb{C}^n$, $F(z,w)$ is a holomorphic function in $\mathbb{C}$. If $F(z,w)$ is not constant, according to Picard Theorem, $F(z,w)$ takes all the values of $\mathbb{C}$ minus at most one point $\pi(z)$. The point $\pi(z)$ is called the exceptional value of $F(z,w)$ in z.

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In the present work we study $\pi(z)$ and

$$\Omega = \{z \in \mathbb{C}^n \mid \pi(z) \text{ exists}\}$$

when $F(z,w)$ is a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w.

1.1. Given $F(z,w)$ and $r_j \in \mathbb{R}^+$ $(j = 1, \dots, i-1)$, if

$$M_F(r_1, \dots, r_{i-1}, z_i, \dots, z_n, t) = \max_{|w|=t,\, |z_j|=r_j} |F(z,w)|$$

we associate:

- The order of $F(z,w)$ in z:

$$\rho(z) = \limsup_{t\to\infty} \frac{\log\log^+ M_F(z_1, \dots, z_n, t)}{\log t}.$$

  It is defined as the order of $w \mapsto F(z,w)$. We say that $F(z,w)$ is of finite order in w if $\rho(z)$ is finite for any z in $\mathbb{C}^n$. According to [4, Theorem 1.41] (see also [4, p. 26]), $F(z,w)$ is of finite order in w if and only if $\rho(z)$ is finite in a non-pluripolar set of $\mathbb{C}^n$.

- The upper order of $F(z,w)$ in the variable w:

$$\rho(r_1, \dots, r_n) = \limsup_{t\to\infty} \frac{\log\log^+ M_F(r_1, \dots, r_n, t)}{\log t}.$$

  It does not depend on r_j $(j = 1, \dots, n)$, and then is a constant, denoted by $\overline{\rho}_w$ [7, p. 110]. Moreover, $\rho(z) \le \overline{\rho}_w$ [7, p. 120].

According to a theorem due to Ronkin [7, Theorem 3.2.1] (proved by Lelong in case $n = 1$), if $F(z,w)$ is of finite order in w, $\overline{\rho}_w$ is finite. In fact, $\rho(z) < \overline{\rho}_w$ in a set of class F_σ.

We will refer through this work to X as an *analytic subset of a domain D in $\mathbb{C}^n$* when, for each point p of D, there are an open neighbourhood U of p in D and a finite family of holomorphic functions on U such that $X \cap U$ is the set of its common zeros. On the other hand, we will say that X is a *local analytic set* in D if the previous U and finite family of holomorphic functions exist for each point p of X (but not necessarily for $p \in D \setminus X$). Clearly, X is an analytic subset of D if and only if, X is a local analytic set and is closed in D. Finally, the sets in $\mathbb{C}^n$ defined by the common zeros of a finite family of holomorphic functions on $\mathbb{C}^n$ are called *globally analytic sets of $\mathbb{C}^n$*.

1.2. In 1963, Nishino [5] studied $\pi(z)$ for a holomorphic $F(z,w)$ in $\mathbb{C} \times \mathbb{C}$ of finite order in w. He showed the following theorem in [5, p. 371]:

Theorem. *(Nishino) Let $F(z,w)$ be a holomorphic function in $\mathbb{C} \times \mathbb{C}$ of finite order in w. If there is a finite accumulation point z_0 of Ω, then there exists a meromorphic function $\alpha(z)$ in $\mathbb{C}$ such that $\pi(z) = \alpha(z)$ except at points z in $\mathbb{C}$ where $\alpha(z)$ has poles or $F(z,w)$ is constant. Moreover, $F(z,w)$ is a polynomial in w when $\alpha(z)$ is ∞.*

Nishino's Theorem implies that

1) The set Ω is a local analytic set in $\mathbb{C}$. Let
$$A = \{z \in \mathbb{C} \,|\, F(z,w) \text{ is constant}\}$$
and consider the Hartogs series expansion of $F(z,w)$ centered at $w = 0$:
$$F(z,w) = \sum_{k=0}^{\infty} F_k(z) w^k,$$
with $F_k(z)$ $(k = 0, 1, \dots)$ holomorphic in $\mathbb{C}$. Since any point z in A holds $\{F_k(z) = 0\}$ $(k = 1, 2 \dots)$, A has no finite accumulation points, and then it is proper analytic subset of $\mathbb{C}$. Then, there is a proper analytic subset E of $\mathbb{C}$ such that either $\Omega \subset E$ (all the points of Ω are isolated) or $\Omega = \mathbb{C} \setminus E$, where E is given by the union of the set of poles of $\alpha(z)$ and A. In both cases, Ω is a local analytic set of $\mathbb{C}$.
2) The graph G_π of $\pi : \Omega \to \mathbb{C}$ is a subset of the graph of a meromorphic function $\alpha(z)$ in $\mathbb{C}^2$ in presence of a finite accumulation point z_0 of Ω. This does not necessarily occur if $F(z,w)$ is not of finite order in w. Let us consider the holomorphic function, [5, p. 367]:
$$F(z,w) = \begin{cases} \frac{e^{ze^w} - 1}{z} & \text{if } \ z \neq 0 \\ \\ e^w & \text{if } \ z = 0 \end{cases}$$
In this case, if $\alpha(z) = -1/z$, $\pi(z) = \alpha(z)$ when $z \neq 0$. However, $\pi(0) = 0$, and $(0,0)$ is in G_π but not in the graph of $\alpha(z)$ in $\mathbb{C}^2$. Moreover, note that in this example $\alpha(0) = \infty$ but $F(0,w)$ is not a polynomial.

In this work we are interested in studying the generalization of Nishino's Theorem to any number of variables:

Problem 1. Consider $n \geq 2$ and a finite accumulation point z_0 of Ω. We want to analyze if there is a neighborhood U of z_0 in $\mathbb{C}^n$ such that $\Omega \cap U$ is a local analytic set in U, and extend Nishino's Theorem when $\Omega \cap U$ is a local analytic set of dimension n, by explicitly determining $F(z,w)$ for it.

We also want to apply the solution of Problem 1 in order to obtain a similar description of Ω as in case $n = 1$:

Problem 2. We want to study if Ω is a local analytic set in $\mathbb{C}^n$, and if there exists a proper analytic subset E of $\mathbb{C}$ such that either $\Omega \subset E$ or $\Omega = \mathbb{C} \setminus E$ when $n \geq 2$.

1.3. Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w. Lelong and Gruman studied the set
$$Z^0 = \{z \in \mathbb{C}^n \,|\, (\text{for all } w)\, F(z,w) \neq 0\,\}.$$
Lelong proved in [3] when $n = 1$ (see also [2, Corollary, p. 688]) that Z^0 or $\mathbb{C} \setminus Z^0$ is discrete. Later, Lelong and Gruman studied the case $n \geq 2$, and proved in [4, Theorem 3.44] the following theorem:

Theorem. *(Lelong-Gruman) Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w, with $n \geq 2$. Consider $A^0 = \{z \in \mathbb{C}^n \,|F(z,w) \equiv 0\}$. Then $Z^0 \cup A^0$ is contained in a proper analytic subset of $\mathbb{C}^n$ unless $Z^0 \cup A^0 = \mathbb{C}^n$.*

Problem 3. We want to study whether Lelong-Gruman Theorem follows from the answer to Problem 2, and thus obtain a generalization of it.

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2. Statement of results

2.1. Main result.

Theorem 1. *Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w, with $n \geq 2$. Consider a finite accumulation point z_0 of Ω. Then there exists a neighborhood U of z_0 in $\mathbb{C}^n$ such that $\Omega \cap U$ is a local analytic set in U and $\pi(z)$ is holomorphic over $\Omega \cap U$. Moreover, there is a proper globally analytic set Δ of $\mathbb{C}^n$ such that either $\Omega \cap U \subset \Delta \cap U$ or $\Omega \cap U = (\mathbb{C}^n \setminus \Delta) \cap U$. In the latter case, when dimension of $\Omega \cap U$ is n, $F(z,w)$ can be explicitly determined: there are holomorphic functions $\xi_i(z)$ $(i = 1,2)$, $v_k(z)$ $(k = 1,\ldots,d \in \mathbb{N}^+)$ and $\eta(z)$ in $\mathbb{C}^n$ such that*

$$F(z,w) = \begin{cases} \dfrac{\xi_2(z)(e^{\xi_1(z)[v_1(z)w+\cdots+v_d(z)w^d]}-1)}{\xi_1(z)} + \eta(z) & \text{if } \ \xi_1(z) \neq 0 \\[2ex] \xi_2(z)[v_1(z)w+\cdots+v_d(z)w^d] + \eta(z) & \text{if } \ \xi_1(z) = 0 \end{cases}$$

Remark 1. We determined $F(z,w)$ in [1] under more restrictive conditions. Concretely, if there is a neighborhood U of z_0 in $\mathbb{C}^n$ contained in Ω and $F(z,w)$ is not constant for any z in $\mathbb{C}^n$.

2.2. Extension of Nishino's Theorem. Theorem 1 solves Problem 1, since it implies that $\Omega \cap U$ is a local analytic set, for a neighborhood U of z_0 in $\mathbb{C}^n$, and allows to obtain explicitly $F(z,w)$ (on $\mathbb{C}^n$) when $\Omega \cap U$ is of dimension n, deducing from it Nishino's Theorem.

Corollary 1. *Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w, with $n \geq 2$. If there exists a neighborhood U of z_0 such that $\Omega \cap U$ is a local analytic set of dimension n then:*

- *a) There is a meromorphic function $\alpha(z)$ in $\mathbb{C}^n$ such that $\pi(z) = \alpha(z)$ on Ω, where $\Omega = \mathbb{C}^n \setminus E$ and E is a proper globally analytic set of $\mathbb{C}^n$. Moreover, $E = A \cup E'$, where A is the set of points z in $\mathbb{C}^n$ such that $F(z,w)$ is constant and E' is the set of poles of $\alpha(z)$.*
- *b) It holds $F(z,w)$ is a polynomial in w if and only if $z \in E$.*
- *c) The order $\rho(z)$ of $F(z,w)$ in w is $d \in \mathbb{N}^+$, except on a globally analytic set of $\mathbb{C}^n$ where $\rho(z) < d$.*

Note that a) in Corollary 1 is a generalization for $n \geq 2$ of 1) and 2) of 1.2. In particular, we obtain Nishino's Theorem (see 1.2) for $n \geq 2$:

Corollary 2. *Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w, with $n \geq 2$. Consider a finite accumulation point z_0 of Ω. If there exists a neighborhood U of z_0 such that $\Omega \cap U$ is a local analytic set of dimension n, then there exists a meromorphic function $\alpha(z)$ in $\mathbb{C}^n$ such that $\pi(z) = \alpha(z)$ except at points z in $\mathbb{C}^n$ where $\alpha(z)$ has poles or $F(z,w)$ is constant. Moreover, $F(z,w)$ is a polynomial in w if $\alpha(z)$ is ∞.*

2.3. Description of Ω. Theorem 1 and Theorem of Ronkin [7, Theorem 3.2.1] (see 1.1) allow to describe Ω as in case $n = 1$ (see 1.2):

Theorem 2. *Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w, with $n \geq 2$. Then Ω is a local analytic set in $\mathbb{C}^n$ and there exists a proper analytic subset E of $\mathbb{C}^n$ such that $\Omega \subset E$ or $\Omega = \mathbb{C}^n \setminus E$.*

Example. Denote $x = (z_1, \ldots, z_{n-1})$ in $\mathbb{C}^{n-1}$ and $z = (x, z_n)$ in $\mathbb{C}^n$. Let $f(x)$ and $g(z)$ be holomorphic functions in $\mathbb{C}^{n-1}$ and $\mathbb{C}^n$, respectively, and define

$$F(z,w) = \begin{cases} \dfrac{e^{g(z)w}}{g(z)} - \dfrac{1}{g(z)} + w\,[z_n - f(x)]\,e^{g(z)w} & \text{if } \; g(z) \neq 0 \\ \\ w\,[1 + z_n - f(x)] & \text{if } \; g(z) = 0 \end{cases}$$

It holds that $\Omega = G_f \cap \{g(z) \neq 0\}$, where G_f is the graph of $f(x)$ in $\mathbb{C}^n$. For each finite accumulation point of Ω there is a neighbourhood U of it such that $\Omega \cap U$ is contained in an analytic set of dimension $n-1$. In this case $\Omega \subset G_f$ is a local analytic set in $\mathbb{C}^n$ of dimension $n-1$.

2.4. Lelong-Gruman Theorem. Consider $n \geq 2$ and define

$$\Omega^0 = \{z \in \mathbb{C}^n \,|\, \pi(z) = 0\} \subset \Omega$$

and $A = \{z \in \mathbb{C}^n \,|F(z,w)$ is constant$\}$. Note that A is a proper analytic subset of $\mathbb{C}^n$: as in case $n = 1$ (1.2), we can expand $F(z,w)$ as Hartogs series centered at $w = 0$,

$$F(z,w) = \sum_{k=0}^{\infty} F_k(z) w^k,$$

with $F_k(z)$ $(k = 0, 1 \ldots)$ holomorphic in $\mathbb{C}^n$. Since A is the intersection of the family $\{F_k(z) = 0\}$ $(k = 1, 2 \ldots)$, we know [6, Corollary 2.1] that A is a proper analytic subset of $\mathbb{C}^n$. Take Z^0 and A^0 defined in 1.3. These sets are related by the following equality

$$Z^0 \cup A^0 = \Omega^0 \cup A.$$

Theorem 2 implies:

Corollary 3. *Let $F(z,w)$ be a holomorphic function in $\mathbb{C}^n \times \mathbb{C}$ of finite order in w, with $n \geq 2$. There exists a proper analytic subset E_0 of $\mathbb{C}^n$ such that $\Omega^0 \subset E_0$ or $\Omega^0 = \mathbb{C}^n$.*

Corollary 3 solves Problem 2 since $Z^0 \cup A^0$ is contained in $E_0 \cup A$ unless $Z^0 \cup A^0 = \mathbb{C}^n$ and Lelong-Gruman Theorem follows from Theorem 2.

3. Proofs

Proof of Theorem 1.

3.1. Consider a finite accumulation point z_0 of Ω. Let $\mathbf{B}(z_0)$ be a ball centered at z_0 of radius $r_0 > 0$. Since $\rho(z)$ is finite for any $z \in \mathbb{C}^n$ (see 1.1), we can assume that $\rho(z)$ on $\mathbf{B}(z_0)$ is bounded by a finite constant ρ_0. Define $\Omega(z_0) = \Omega \cap \mathbf{B}(z_0)$. Note that z_0 can not be in $\Omega(z_0)$ if $z_0 \in \partial\Omega(z_0)$.

Take z in $\Omega(z_0)$. It follows by Hadamard's Theorem that

$$F(z,w) - \pi(z) = e^{g(z,w)},$$

with $g(z,w)$ a polynomial in w of degree $\rho(z)$. Therefore $\rho(z) \in \mathbb{N}^+$ on $\Omega(z_0)$ and $\rho_0 > 0$ (see, [8, 8.24 and 8.26]).

Consider $\eta(z) = F(z,0)$ and define

$$\tilde{F}(z,w) = F(z,w) - \eta(z).$$

If z in $\Omega(z_0)$, then $\pi(z) - \eta(z)$ is an exceptional value of $\tilde{F}(z,w)$. Note that $\pi(z) - \eta(z)$ is not zero since $\tilde{F}(z,0)$ is identically zero. Define $C_0(z)$ such that

$$-1/C_0(z) = \pi(z) - \eta(z).$$

We can write

$$\tilde{F}(z,w) = \frac{e^{g(z,w)}}{C_0(z)} - \frac{1}{C_0(z)},$$

with $g(z,w) = C_1(z)w + C_2(z)w^2 + \dots$ and $C_k(z)$ complex numbers for $k \geq 1$. Note that $C_k(z) = 0$ if k is an integer $> \rho(z)$.

3.2. Given z in $\Omega(z_0)$, take successive partial derivatives with respect to w of $\tilde{F}(z,w)$ at $w = 0$. Because $\tilde{F}(z,0)$ is equal to zero, we obtain [5]:

1) $C_1(z) = C_0(z) f_1(z)$, with $f_1(z) = \frac{\partial \tilde{F}}{\partial w}(z,0)$,

2) $C_2(z) = C_0(z) f_2(z) - \frac{1}{2}[C_1(z)]^2$, with $f_2(z) = \frac{1}{2!}\frac{\partial^2 \tilde{F}}{\partial w^2}(z,0)$,
and thus the general case:

k) $C_k(z) = C_0(z) f_k(z) - Q_k[C_1(z), \dots, C_{k-1}(z)]$ $(k = 2,3,\dots)$ where Q_k is a polynomial of $k-1$ variables with coefficients in $\mathbb{Q}$, and

$$f_k(z) = \frac{1}{k!}\frac{\partial^k \tilde{F}}{\partial w^k}(z,0).$$

Then, if z is in $\Omega(z_0)$ we conclude [5, p. 370]:

(*i*) $C_k(z)$ is expressed as a polynomial in the variable $C_0(z)$ with coefficients given by *holomorphic functions* $f_k(z)$ $(k = 1, 2, \dots)$ *in* $\mathbb{C}^n$ evaluated in z, for each positive integer k. Then, *there exist polynomials* $R_k(z, w)$ $(k = 1, 2, \dots)$ *in* w *whose coefficients are holomorphic functions in* $\mathbb{C}^n$ *such that*

$$C_k(z) = R_k(z, C_0(z)).$$

(*ii*) Since $\rho(z) \le \rho_0$, if $d = [\rho_0]$, it holds $C_k(z) = 0$ for each integer $k \ge d+1$.

The most important consequence of the above points is the following lemma:

Lemma 1. *Fixed a point* (z', w_1) *in* $\mathbf{B}(z_0) \times \mathbb{C}$, *with* $w_1 \neq 0$, *then* $R_k(z', w_1) = 0$ *for* $k \ge d+1$ *and infinitely many* $f_k(z')$ $(k = 1, 2 \dots)$ *are different from zero if and only if* $\tilde{F}(z', w)$ *has an exceptional value* $-1/w_1$.

Proof. If $R_k(z', w_1) = 0$ for $k \ge d+1$, by definition of $R_k(z, w)$ $(k = 1, 2 \dots)$ given in (*i*), and 1), 2) and *k*) in 3.2, one can obtain $f_k(z')$ recursively in terms of $R_k(z', w_1)$ $(k = 1, 2 \dots)$. If infinitely many $f_k(z')$ $(k = 1, 2 \dots)$ are $\neq 0$, they determine

$$\tilde{F}(z', w) = \frac{e^{g(z', w)}}{w_1} - \frac{1}{w_1},$$

with $g(z', w) = R_1(z', w_1)w + R_2(z', w_1)w^2 + \cdots + R_d(z', w_1)w^d$. Therefore $\tilde{F}(z', w)$ has an exceptional value $-1/w_1$. The other implication is clear from 3.1, 3.2. □

Remark 2. After Lemma 1, z is in $\Omega(z_0)$ if and only if there exists $w \neq 0$ such that $R_k(z, w) = 0$ for $k \ge d+1$ and infinitely many $f_k(z)$ are not zero. In this case, $\pi(z) = \eta(z) - 1/w$.

Lemma 2. *If* A^p *is the set of points* z *in* $\mathbb{C}^n$ *such that* $F(z, w)$ *is a polynomial in* w, *then* $A^p \cap \mathbf{B}(z_0)$ *is an analytic subset of* $\mathbf{B}(z_0)$.

Proof. Consider the expansion of $F(z, w)$ as Hartogs series centered at $w = 0$ (see 2.4). Define the family of subsets $U_k \subset A^p$ $(k = 0, 1, \dots)$ of points z in $\mathbf{B}(z_0)$ such that $F(z, w)$ is a polynomial of degree at most k. It holds that U_k is the intersection of the family of analytic subsets of $\mathbf{B}(z_0)$: $\{F_j(z) = 0\} \cap \mathbf{B}(z_0)$ $(j = k+1, \dots)$, and then an analytic subset of $\mathbf{B}(z_0)$ [6, Corollary 2.1]. It is clear that $A^p = \cup_{k=0}^{\infty} U_k$ and that $U_k \subset U_{k+1}$ $(k = 0, 1, \dots)$. Since the dimension of U_k is $0 < d_k \le n-1$ and $d_k \le d_{k+1}$ $(k = 0, 1, \dots)$, there is $k_0 \in \mathbb{N}$ such that $U_k = U_{k_0}$ $(k = k_0 + 1, \dots)$ and $A^p = U_{k_0}$ [6, Remark 2.10]. □

3.3.

Proposition 1. *Let* $F(z, w)$ *be a holomorphic function in* $\mathbb{C}^n \times \mathbb{C}$ *of finite order in* w, *with* $n \ge 2$. *Consider a finite accumulation point* z_0 *of* Ω. *Then there exists a neighborhood* U *of* z_0 *in* $\mathbb{C}^n$ *such that* $\Omega \cap U$ *is a local analytic set in* U *and* $\pi(z)$ *is holomorphic over* $\Omega \cap U$. *Moreover, there is a proper globally analytic set* Δ *of* $\mathbb{C}^n$ *such that either* $\Omega \cap U \subset \Delta \cap U$ *or* $\Omega \cap U = (\mathbb{C}^n \setminus \Delta) \cap U$.

Proof. Take $U = \mathbf{B}(z_0)$. Consider the family $\{S_j\}$ $(j = 1, 2 \dots)$ of globally analytic sets of $\mathbb{C}^{n+1}$ defined by $\{R_{d+j}(z, w) = 0\}$ where $R_{d+j}(z, w)$ are as in 3.2. The points (z, w) in $\mathbb{C}^{n+1}$ such that z in $\Omega(z_0)$ and $w = C_0(z)$ define a subset $L \subset S_j$ $(j = 1, 2 \dots)$. Then the intersection of $\{S_j\}$ defines a globally analytic set S of $\mathbb{C}^{n+1}$ containing L [6, Corollary 2.1].

Let $\tilde{z} \in \Omega(z_0)$. By Lemma 2, there is a ball $\mathbf{B}(\tilde{z}) \subset \mathbb{C}^n$ of center $\tilde{z}$ contained in $\mathbf{B}(z_0)$ such that $\mathbf{B}(\tilde{z}) \cap A^p$ is empty. Then, for any $z \in \mathbf{B}(\tilde{z})$, infinitely many $f_k(z)$ $(k = 1, 2 \dots)$ are $\neq 0$. Consider $S^* = S \setminus \{w = 0\}$, $\tilde{L} = L \cap (\mathbf{B}(\tilde{z}) \times \mathbb{C})$, and the projection $\Pi_1 : \mathbb{C}^{n+1} \to \mathbb{C}^n$, $\Pi_1(z, w) = z$. After Lemma 1 and Remark 2, it follows that

$$\tilde{L} = S^* \cap (\mathbf{B}(\tilde{z}) \times \mathbb{C})$$

and $\Pi_1(\tilde{L}) = \Omega(z_0) \cap \mathbf{B}(\tilde{z})$. Since $\Pi_{1|S} : S \to \Pi_1(S)$ is a proper map, then $\Omega(z_0) \cap \mathbf{B}(\tilde{z})$ is a local analytic set. It proves that $\Omega \cap U$ is a local analytic set in U. It follows by [6, Remark 2.8] that $C_0(z)$ and $\pi(z) = -1/C_0(z)$ are holomorphic on $\Omega(z_0) \cap \mathbf{B}(\tilde{z})$, and $\pi(z)$ is holomorphic on $\Omega \cap U$.

Note that the above analysis implies that

$$L = S^* \cap ((\mathbf{B}(z_0) \setminus A^p) \times \mathbb{C})$$

is the graph of holomorphic function $C_0(z)$ on $\Omega(z_0)$.

Due to [6, Remark 2.10], S must be the intersection of a finite family $\{S_{r_j}\}$ $(j = 1, \dots, q)$ where r_j $(j = 1, \dots, q)$ are different integers in $\mathbb{N}^+$. Explicitly, according to 3.2, each S_{r_j} $(j = 1, \dots, q)$ is defined by the zeros of a polynomial in w with holomorphic coefficients:

$$R_{d+r_j}(z, w) = A_0^j(z) + A_1^j(z)w + \cdots + A_{l_j}^j(z)w^{l_j},$$

with $A_h^j(z)$ $(h = 0, \dots, l_j \in \mathbb{N})$ holomorphic in $\mathbb{C}^n$, where $A_0^j(z)$ is assumed to be not identically zero since L does not intersect $\{w = 0\}$.

There are two possibilities:

1) There exists $j_0 \in \{1, \dots, q\}$ with $l_{j_0} > 1$. Consider the discriminant $d_{j_0}(z)$ of $R_{d+r_j}(z, w)$ with respect to w. In this case, for $z \in \Omega(z_0)$, it holds

$$A_{l_{j_0}}^{j_0}(z) \cdot d_{j_0}(z) = 0,$$

since $A_{l_{j_0}}^{j_0}(z) \neq 0$ implies $d_{j_0}(z) = 0$. Otherwise the l_{j_0} points: (z, w_i) $(i = 1, \dots, l_{j_0})$ are in L, which is not possible because L is the graph of a holomorphic function. Since $\Omega(z_0) \subset \mathcal{E}_{j_0} \cap U$, with

$$\mathcal{E}_{j_0} = \{A_{l_{j_0}}^{j_0}(z) = 0\} \cup \{d_{j_0}(z) = 0\},$$

then $\Omega(z_0) \subset \Delta \cap U$, where Δ is the intersection of the family of sets $\{\mathcal{E}_{j_0}\}$, with $j_0 \in \{1, \dots, q\}$ and $l_{j_0} > 1$.

2) For all $j \in \{1, \dots, q\}$, $l_j = 1$. In this case S is the intersection of

$$S_j = \{\, A_0^j(z) + A_1^j(z)w = 0\} \ (j = 1, \dots, q).$$

Take $\xi_{1j}(z)$, $\xi_{2j}(z)$ $(j = 1, \ldots, q)$ holomorphic functions in $\mathbb{C}^n$ relatively prime at any z such that $A_0^j(z) + A_1^j(z)w = p^j(z)[\xi_{1j}(z)(z) - \xi_{2j}(z)w]$, with $p^j(z)$ holomorphic in $\mathbb{C}^n$. Consider

$$\Gamma_j(z) = \begin{vmatrix} A_0^1(z) & A_0^j(z) \\ A_1^1(z) & A_1^j(z) \end{vmatrix} \ (j = 1, \ldots, q).$$

2a) (All S_j are dependent) If $q = 1$, or $q > 1$ and $\Gamma_j(z) \equiv 0$ $(j = 2, \ldots, q)$: $\Omega(z_0) = (\mathbb{C}^n \setminus \Delta) \cap U$, with $\Delta = \Delta_1 \cup \Delta_2$, where

$$\Delta_1 = \{\xi_{11}(z) = 0\} \cup \{\xi_{21}(z) = 0\}$$

and Δ_2 is given by the intersection of $\{p^j(z) = 0\}$ $(j = 1, \ldots, q)$.

2b) If $q > 1$ and there is $j_0 \in \{2, \ldots, q\}$ such that $\Gamma_{j_0}(z)$ is not identically zero, $\Delta_{j_0} = \{\Gamma_{j_0}(z) = 0\}$ defines a proper globally analytic of $\mathbb{C}^n$ such that $\Omega(z_0) \subset \Delta_{j_0} \cap U$. Then $\Omega(z_0) \subset \Delta \cap U$, where Δ is the intersection of sets Δ_{j_0}, with $j_0 \in \{2, \ldots, q\}$ such that $\Gamma_{j_0}(z) \not\equiv 0$.

It finishes the proof of Proposition 1. □

3.4. Let us suppose that dimension of $\Omega(z_0)$ is n. It is clear that 2a) of 3.3 holds and $\Omega(z_0) = (\mathbb{C}^n \setminus \Delta) \cap U = \mathbf{B}(z_0) \setminus \Delta$. Denote $\xi_{11}(z)$ and $\xi_{21}(z)$, respectively, by $\xi_1(z)$ and $\xi_2(z)$. Consider

$$\xi(z) = \frac{\xi_1(z)}{\xi_2(z)} = -\frac{A_0^1(z)}{A_1^1(z)}.$$

The meromorphic function $-1/\xi(z)$ restricted to $\mathbf{B}(z_0) \setminus \Delta$ is equal to $-1/C_0(z)$. If we substitute $C_0(z)$ by $\xi(z)$ (meromorphic function in $\mathbb{C}^n$!) in the definitions of $C_k(z)$ of 3.2, taking $C_k(z) = R_k(z, \xi(z))$ $(k = 1, 2, \ldots)$ we obtain a function

$$G(z, w) = \frac{e^{C_1(z)w + \cdots + C_d(z)w^d}}{\xi(z)} - \frac{1}{\xi(z)}.$$

Note that $G(z, w)$ is holomorphic in $(\mathbb{C}^n \setminus \Delta) \times \mathbb{C}$ and coincides with $\tilde{F}(z, w)$ in $(\mathbf{B}(z_0) \setminus \Delta) \times \mathbb{C}$ (see 3.2). Then, $G(z, w)$ is holomorphic in $\mathbb{C}^{n+1}$ and equals to $\tilde{F}(z, w)$. Therefore, if $g(z, w) = C_1(z)w + \cdots + C_d(z)w^d$, $g(z, w)/\xi(z)$ is holomorphic in $\mathbb{C}^{n+1}$ and $g(z, w) = \xi(z)[u_1(z)w + \cdots + u_d(z)w^d]$ with $u_k(z)$ holomorphic in $\mathbb{C}^n$ $(k = 1, 2, \ldots, d)$.

Explicitly,

$$\begin{aligned}\tilde{F}(z, w) = \frac{1}{1!}[u_1(z)w + \cdots + u_d(z)w^d] + \\ + \frac{\xi(z)}{2!}[u_1(z)w + \cdots + u_d(z)w^d]^2 + \\ + \frac{\xi(z)^2}{3!}[u_1(z)w + \cdots + u_d(z)w^d]^3 + \cdots\end{aligned}$$

3.5. It holds that

$$\{z \in \mathbb{C}^n \,|\tilde{F}(z,w) \text{ is constant}\} = \{z \in \mathbb{C}^n \,|u_1(z) = \cdots = u_d(z) = 0\}.$$

First we treat the case where $\xi(z)$ is holomorphic.

Consider z' in $\mathbb{C}^n$ such that $\tilde{F}(z',w)$ is constant. If $\xi(z') = 0$, according to the expansion of $\tilde{F}(z,w)$ in 3.4, it holds if and only if $u_1(z')w + \cdots + u_d(z')w^d = 0$ for any $w \in \mathbb{C}$, or equivalently if $u_k(z') = 0$ $(k = 1, \ldots, d)$. If $\xi(z') \neq 0$, as

$$\xi(z)\tilde{F}(z,w) + 1 = e^{\xi(z)\cdot[u_1(z)w+\cdots+u_d(z)w^d]},$$

in the same way it is equivalent to $u_k(z') = 0$ $(k = 1, \ldots, d)$.

Now, we treat the case where $\xi(z)$ is not holomorphic.

Consider a point z' in $\mathbb{C}^n$ such that $\xi_1(z') \neq 0$ and $\xi_2(z') = 0$. Assume that there is w_1 such that $[u_1(z')w_1 + \cdots + u_d(z'){w_1}^d] \neq 0$ and take a ball $\mathbf{B}(z',w_1)$ in $\mathbb{C}^{n+1}$ centered at (z',w_1) of radius $r_1 > 0$ such that $\overline{\mathbf{B}}(z',w_1) \cap \{(z,w)|\xi_1(z) = 0\}$ is empty. Take a line ℓ in $\mathbb{C}^{n+1}$ passing through (z',w_1) and consider $\ell_0 = \ell \cap \overline{\mathbf{B}}(z',w_1)$. We can assume by [6, Lemma 2.8] that

$$\ell_0 \cap \{(z,w)|\xi_2(z) = 0\} = \{(z',w_1)\}.$$

Suppose that ℓ_0 is defined by $\{(z,w) = \lambda_0 t + (z',w_1)\}$, for a fixed $\lambda_0 \in \mathbb{C}^{n+1}$ and t in a disk $\mathbb{D}_\epsilon$ in $\mathbb{C}$ of center $t = 0$ and radius sufficiently small $\epsilon > 0$. It follows by the above expansion of $\tilde{F}(z,w)$ in 3.4 that $\tilde{F}(z,w)$ over L_0 is of the form $(e^{g(t)} - 1)/g(t)$, where $g(t)$ is holomorphic on $\mathbb{D}_\epsilon \setminus \{0\}$, and with a pole of positive order at $t = 0$. It implies the existence of an essential singularity of $\tilde{F}(z,w)$ over ℓ_0 at (z',w_1) and contradicts that $\tilde{F}(z,w)$ is holomorphic. Then, if $\xi_1(z') \neq 0$ and $\xi_2(z') = 0$ necessarily $[u_1(z')w + \cdots + u_d(z')w^d] = 0$, for any w in $\mathbb{C}$, and $u_k(z') = 0$ $(k = 1, \ldots, d)$. It follows that

$$\{\xi_2(z) = 0\} \subset \{u_1(z) = \cdots = u_d(z) = 0\}.$$

Then $u_k(z) = v_k(z)\xi_2(z)$ $(k = 1, \ldots, d)$, with $v_k(z)$ holomorphic in $\mathbb{C}^n$, and

$$\begin{aligned}\tilde{F}(z,w) = \frac{1}{1!}\xi_2(z)[v_1(z)w + \cdots + v_d(z)w^d]+ \\ + \frac{\xi_1(z)\xi_2(z)}{2!}[v_1(z)w + \cdots + v_d(z)w^d]^2 + \\ + \frac{\xi_1(z)^2\xi_2(z)}{3!}[v_1(z)w + \cdots + v_d(z)w^d]^3 + \cdots\end{aligned}$$

From this expression, it is clear that a point z in $\mathbb{C}^n$ verifies $\tilde{F}(z,w)$ is constant if and only if $u_k(z) = 0$ $(k = 1, \ldots, d)$, and $F(z,w) = \tilde{F}(z,w) + \eta(z)$ is as in the statement of Theorem.

Proof of Corollary 1. It is enough to analyze the explicit expression of $F(z,w)$ obtained in the statement of Theorem 1. For a), we take

$$\alpha(z) = -\frac{\xi_2(z)}{\xi_1(z)} + \eta(z),$$

$A = \{\xi_2(z) = 0\} \cup \{v_1(z) = \cdots = v_d(z) = 0\}$, and $E' = \{\xi_1(z) = 0\}$. The point *b*) is clear. For *c*), we see $\rho(z)$ is d, except on $E \cup \{v_d(z) = 0\}$ where is $< d$.

Proof of Corollary 2. It follows directly from *a*), *b*) of Corollary 1.

Proof of Theorem 2. Consider Ω_1 and Ω_2, respectively, the set of finite accumulation points of Ω and the set of isolated points of Ω. If Ω_1 is empty, $\Omega = \Omega_2$ is closed and discrete in $\mathbb{C}^n$, and then it defines a proper analytic subset E of $\mathbb{C}^n$. Suppose $z_0 \in \Omega_1$. If there exists a neighborhood U of z_0 such that $\Omega \cap U$ is a local analytic set of dimension n, according to *a*) of Corollary 1, there exists a proper analytic subset E of $\mathbb{C}^n$ such that $\Omega = \mathbb{C}^n \setminus E$ (note that in this case, Ω_2 is empty). If there exists a neighborhood U of z_0 such that $\Omega \cap U$ is a local analytic set of dimension $< n$, according to the proof of Theorem 1, concretely, Proposition 1, there exist a neighborhood U of z_0 in $\mathbb{C}^n$ and a proper globally analytic set Δ of $\mathbb{C}^n$ such that $\Omega \cap U \subset \Delta \cap U$. Theorem of Ronkin (see 1.1) allows to take $\rho_0 = \overline{\rho}_w$ independently of the point z_0, and then conclude that Δ is the same set for all the finite accumulation points of Ω. Then, it is enough to define the proper analytic subset $E = \Delta \cup \Omega_2$ of $\mathbb{C}^n$ to obtain $\Omega \subset E$.

Proof of Corollary 3. Since $\Omega^0 \subset \Omega$, if dimension of Ω is $< n$, the proof follows from Theorem 2 taking $E_0 = E$. If dimension of Ω is n, according to *a*) of Corollary 1, Ω^0 is defined by the analytic subset E_0 of $\mathbb{C}^n$ given by the zeros of $\alpha(z)$. Then E_0 is proper if $\alpha(z)$ is not identically zero or $\mathbb{C}^n$ otherwise.

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